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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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such that the former decreases and the latter increases continuously in the given interval. Hence, by the help of Tchebycheff's inequality,* we get

$$\frac{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\,F_1(\alpha,\beta,\beta',\gamma,x,y)$$

$$<\int_0^1 u^{\alpha-1}\,du\int_0^1(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}du$$

i.e., $$<\frac{F_1(1,\beta,\beta',\gamma-\alpha+1,x,y)}{\alpha(\gamma-\alpha)}.$$

Therefore

$$F_1(\alpha,\beta,\beta',\gamma,x,y)<\frac{\Gamma(\gamma)}{\Gamma(\alpha+1)\,\Gamma(\gamma-\alpha+1)}\times$$

$$F_1(1,\beta,\beta',\gamma-\alpha+1,x,y). \quad \ldots \quad (1)$$

2. If we further suppose that β and β' are such that

$$(1-ux)^{-\beta}(1-uy)^{-\beta'}$$

is an increasing function of u in the interval $0\leqslant u\leqslant 1$, we get by a further application of Tchebycheff's inequality

$$\frac{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\,F_1(\alpha,\beta,\beta',\gamma,x,y)$$

$$<\int_0^1 u^{\alpha-1}\,du\int_0^1(1-u)^{\gamma-\alpha-1}\,du\int_0^1(1-ux)^{-\beta}(1-uy)^{-\beta'}du$$

i.e., $$<\frac{F_1(1,\beta,\beta',2,x,y)}{\alpha(\gamma-\alpha)}.$$

Hence we have a stronger inequality

$$F_1(\alpha,\beta,\beta',\gamma,x,y)<\frac{\Gamma(\gamma)}{\Gamma(\alpha+1)\,\Gamma(\gamma-\alpha+1)}\cdot F_1(1,\beta,\beta',2,x,y)$$

$$\ldots \quad (2)$$

3. If, however, β' only be restricted in such a way that $(1-uy)^{-\beta'}$ is an increasing function of u, we obtain, in a similar manner,

$$\frac{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\,F_1(\alpha,\beta,\beta',\gamma,x,y)$$

$$<\int_0^1 u^{\alpha-1}\,du\int_0^1(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}\,du\int_0^1(1-uy)^{-\beta'}du$$

* See Hermite—*Cours professé pendant le 2e Semestre*, 1881-82.

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$$i.e., \quad < \frac{\left\{1-(1-y)^{-\beta'+1}\right\}}{y\alpha(1-\beta')(\gamma-\alpha)} F(\beta, 1; \gamma-\alpha+1; x),$$

where F stands for the usual hyper-geometric series of a single variable.

Therefore we arrive at another inequality, stronger than (1),

$$F_1(\alpha, \beta, \beta', \gamma, x, y)$$

$$< \frac{\{1-(1-y)^{-\beta'+1}\}\,\Gamma(\gamma)}{y(1-\beta')\Gamma(\alpha+1)\Gamma(\gamma-\alpha+1)} F(\beta, 1; \gamma-\alpha+1: x).$$

4. We shall now give three particular cases of (1), (2) and (3), in each of which the second and third inequalities are stronger than the first.

NOTE 1. When $y = x$, we know that

$$F_1(\alpha, \beta, \beta', \gamma, x, x) = F(\alpha, \beta+\beta'; \gamma; x).$$

The relations (1), (2) and (3), in this case, reduce to

$$F(\alpha, \beta+\beta'; \gamma; x) <$$
$$\frac{\Gamma(\gamma)}{\Gamma(\alpha+1)\Gamma(\gamma-\alpha+1)} F(1, \beta+\beta'; \gamma-\alpha+1; x),$$

$$F(\alpha, \beta+\beta'; \gamma; x) <$$
$$\frac{\Gamma(\gamma)}{\Gamma(\alpha+1)\Gamma(\gamma-\alpha+1)} F(1, \beta+\beta'; 2; x),$$

and

$$F(\alpha, \beta+\beta'; \gamma; x) <$$
$$\frac{\{1-(1-x)^{-\beta'+1}\}\,\Gamma(\gamma)}{x(1-\beta')\Gamma(\alpha+1)\Gamma(\gamma-\alpha+1)} F(\beta, 1; \gamma-\alpha+1; x).$$

NOTE 2. When $y = 1$,

$$F_1(\alpha, \beta, \beta', \gamma, x, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta')} F(\alpha; \beta; \gamma-\beta': x),$$

$$[R(\gamma-\alpha-\beta') > 0]$$

In this case, with the further restriction $\beta' < 1$, we obtain from (1), (2) and (3), the following inequalities :—

$$F(\alpha, \beta ; \gamma-\beta' ; x) <$$
$$\frac{\Gamma(\gamma-\beta')}{\Gamma(\alpha+1)\,\Gamma(\gamma-\alpha-\beta'+1)}\, F(1, \beta ; \gamma-\alpha-\beta'+1 ; x),$$

$$F(\alpha, \beta ; \gamma-\beta' ; x) <$$
$$\frac{\Gamma(\gamma-\beta')}{(1-\beta')(\gamma-\alpha)\,\Gamma(\alpha+1)\,\Gamma(\gamma-\alpha-\beta')}\, F(1, \beta ; 2-\beta' ; x),$$

and

$$F(\alpha, \beta ; \gamma-\beta' ; x) <$$
$$\frac{\Gamma(\gamma-\beta')}{(1-\beta')(\gamma-\alpha)\,\Gamma(\alpha+1)\,\Gamma(\gamma-\alpha-\beta')}\, F(\beta, 1; \gamma-\alpha+1 ; x).$$

NOTE 3. When $x = y = 1$,

$$F_1(\alpha, \beta, \beta', \gamma, 1, 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta-\beta')}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta-\beta')};$$

$$[R(\gamma-\alpha-\beta-\beta') > 0]$$

Hence, under the additional restriction $\beta + \beta' < 1$, we obtain from (1), (2), and (3) the following inequalities satisfied by the Gamma-function :—

$$\Gamma(\gamma-\beta-\beta') > \Gamma(\alpha+1)\,\Gamma(\gamma-\alpha-\beta-\beta'+1) ; \ldots \quad (4)$$

$$\Gamma(\gamma-\beta-\beta') > (1-\beta-\beta')(\gamma-\alpha)\,\Gamma(\alpha+1)\Gamma(\gamma-\alpha-\beta-\beta');$$

and $\Gamma(\gamma-\beta-\beta') > (1-\beta')(\gamma-\alpha-\beta)\,\Gamma(\alpha+1)\,\Gamma(\gamma-\alpha-\beta-\beta')$.

By putting $\alpha = p$, $\gamma-\alpha-\beta-\beta' = q$, the ine quality (4) assumes the known form*

$$\Gamma(p+1)\,\Gamma(q+1) < \Gamma(p+q).$$

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* Pramathanath Mitra: "On some Integral Inequalities," *Bulletin of the Calcutta Math. Soc.*, Vol. XXII, Nos. 2 and 3 (1930), p. 139.

ON SOME PROPERTIES OF THE ZETA FUNCTION.*

BY V. THIRUVENKATACHARYA, M.A., L.T.

1. Introduction.—This paper is based on the 'Memoire sur les Polynomes de Bernoulli' by Nörlund which appeared in the *Acta Mathematica*, Vol 43, (1922), pp. 121—196. Nörlund has defined Bernoulli's function $B_\gamma(x)$ to be the solution of the finite difference equation $f_\gamma(x+1) - f_\gamma(x) = \gamma x^{\gamma-1}$ for positive integral values of γ. By taking γ negative, I derive in this paper the relationship between $B_\gamma\ [= B_\gamma(1)]$ and the Reimann's Zeta function $\zeta(\gamma)$. A few properties of the zeta function are thus derived and with them the theorem, which shall be called the addition theorem for the zeta function. The formula of Ramanujan connecting $B_{-\gamma}$ and $\zeta(\gamma)$ is also obtained.

2. Let the function $B_{-\gamma}(x)$ be called a Bernoulli's function if it satisfies the function equation $f(x+1) - f(x) = -\gamma x^{-\gamma-1}$ where γ is a positive number. Let Δ denote the operator such that $\Delta f(x) = f(x+1) - f(x)$. Let D_x denote differentiation with respect to x and D_x^p differentiation with respect to x taken successively p times.

Now $$\Delta B_{-\gamma}(x) = -\gamma x^{-\gamma-1} \quad \ldots \quad \ldots \quad (2.1)$$

By differentiation we have

$$D_x \Delta B_{-\gamma}(x) = \gamma(\gamma+1) x^{-\gamma-2} = -\gamma \Delta B_{-\gamma-1}(x). \quad (2.2)$$

$$\therefore \quad D_x B_{-\gamma}(x) = -\gamma B_{-\gamma-1}(x) \quad \ldots \quad \ldots \quad (2.3)$$

$$\therefore \quad D_x^p B_{-\gamma}(x) = (-1)^p \gamma(\gamma+1)\ldots(\gamma+p-1) B_{-\gamma-p}(x) \quad \ldots \quad (24)$$

From (2.3) by integration we have

$$\int_x^y B_{-\gamma-1}(x)\,dx = -\frac{1}{\gamma}\left\{B_{-\gamma}(y) - B_{-\gamma}(x)\right\} \quad \ldots \quad (2.5)$$

---

* Presented to the Indian Mathematical Conference held at Trivandrum in April 1931.

$$\therefore \int_1^{p+1} B_{-\gamma-1}(x)\, dx = -\frac{1}{\gamma}\left\{ B_{-\gamma}(p+1) - B_{-\gamma}(1) \right\} = \sum_{x=1}^{p} x^{-\gamma-1} \qquad \dots (2.51)$$

Let us assume that $B_{-\gamma}(1) = B_{-\gamma}$. ... (2.6)

3. We shall prove that $B_{-\gamma}(x) \to 0$ as $x \to \infty$.

By definition $B_{-\gamma}(x+1) - B_{-\gamma}(x) = -\gamma x^{-\gamma-1}$,

$$\text{Now } \underset{x\to\infty}{\text{Lt}} \frac{B_{-\gamma}(x)}{x} = \underset{x\to\infty}{\text{Lt}} \frac{B_{-\gamma}(x+1) - B_{-\gamma}(x)}{(x+1)-x}$$

$$= \underset{x\to\infty}{\text{Lt}} -\gamma \,.\, x^{-\gamma-1}.$$

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$$\therefore \underset{x\to\infty}{\text{Lt}} B_{-\gamma}(x) = \underset{x\to\infty}{\text{Lt}} (-\gamma x^{-\gamma}) = 0 \text{ as } \gamma \text{ is positive.} \qquad (3.1)$$

4. Making $p \to \infty$ in (2.51), we get from (2.51) and (3.1),

$$\int_1^{\infty} B_{-\gamma-1}(x)\, dx = \frac{B_{-\gamma}}{\gamma} = \sum_{x=1}^{\infty} \frac{1}{x^{1+\gamma}} = \zeta(\gamma+1) \quad \dots (4.1)$$

Ramanujan has been able to derive this result by a different method, which I found in his manuscript note-books kept in the Madras University Library.

Since $B_{-\gamma}(x+h)$ is an analytic function of x, we can apply Taylor's theorem for its expansion as a power series in h. With the help of (2.4) we have

$$B_{-\gamma}(x+h) = B_{-\gamma}(x) - \gamma . h . B_{-\gamma-1}(x) + \frac{\gamma(\gamma+1)}{1.2} h^2 B_{-\gamma-2}(x) + \dots \qquad \dots (4.2)$$

which is an infinite series.

Let $h = 1$ and we have

$$-\gamma . x^{-\gamma-1} = -\gamma . B_{-\gamma-1}(x) + \frac{\gamma.(\gamma+1)}{1.2} B_{-\gamma-2}(x) + \dots\dots \qquad (4.3)$$

This is a recurrence formula.

Putting $x=1$, we have a formula for Reimann's Zeta function

$$1 = (\gamma + 1)\,\zeta(\gamma + 2) + \frac{(\gamma + 1)(\gamma + 2)}{1.2}\,\zeta(\gamma+3) + \ldots \quad (4.4)$$

This is true for $\gamma \geqslant 0$.

For example when $\gamma = 0$, we have the interesting result

$$1 = \zeta(2) - \zeta(3) + \zeta(4) - \zeta(5) + \ldots \ldots$$

This can be easily verified by writing out the infinite series for $\zeta(s)$ and summing by columns.

We get a more general result than (4.4) in what follows.

Making $y \rightarrow \infty$ in (2.5) we have

$$\int_x^\infty B_{-\gamma-1}(x)\,dx = -\frac{B_{-\gamma}(x)}{\gamma} = \zeta(\gamma + 1, x) \quad \ldots \quad (4.5)$$

Substituting for $B_{-\gamma-1}(x)$ in the integrand of (4.5) we have

$$(\gamma + 1)\int_1^\infty \int \zeta(\gamma + 2, x) = \zeta(\gamma + 1, x) \quad \ldots \quad (4.6)$$

a result which can be directly verified. From (4.3) we have by the substitution

$$\frac{B_{-\gamma-s}(x)}{\gamma + s} = \zeta(\gamma + s + 1, x)$$

and after a slight simplification a recurrence formula for the generalized zeta function

$$\frac{1}{x^{\gamma+1}} = (\gamma + 1)\,\zeta(\gamma + 2, x) - \frac{(\gamma + 1)(\gamma + 2)}{1.2}\,\zeta(\gamma + 3, x)$$

$$+ \frac{(\gamma + 1)(\gamma + 2)(\gamma + 3)}{1.2.3}\,\zeta(\gamma + 4, x) - + \ldots \quad (4.7)$$

5. We can show that our formula (2.51) is consistent with that of Nörlund when γ is changed into $-\gamma$.

$$\sum_1^p x^{-\gamma-1} = \frac{1}{\gamma}\left\{B_{-\gamma}(p+1) - B_{-\gamma}(1)\right\} = \zeta(\gamma+1) - \frac{B_{-\gamma}(p+1)}{\gamma}.$$

Now change γ into $-\gamma$; then

$$\sum_{1}^{p} x^{\gamma-1} = \zeta(-\gamma+1) + \frac{B_\gamma(p+1)}{\gamma}.$$

We know that

$$\zeta(-2m) = 0 \text{ and } \zeta(1-2m) = (-)^m \frac{B_m}{2m} \text{ and } \zeta(0) = -\tfrac{1}{2}.$$

Considering odd and even values of γ say $2m-1$ and $2m$, we have

$$\Sigma\, x^{2m-1} = \zeta(1-2m) + \frac{B_{2m}(p+1)}{2m} = \frac{B_{2m}(p+1) + (-1)^m B_m}{2m};$$

$$\Sigma\, x^{2m} = \zeta(-2m) + \frac{B_{2m+1}(p+1)}{2m+1} = \frac{B_{2m+1}(p+1)}{2m+1}.$$

These two correspond to Nörlund's formula, if we remember that his $B_{2m+1} = 0$ and $B_{2m} = (-)^{m-1} B_m$.

6. Addition Theorem.—Let us consider the function

$$f(x) = \sum_{s=0}^{n-1} B_{-\gamma}\left(x + \frac{s}{n}\right)$$

n being any positive integer whatever. Then we have

$$f\left(x + \frac{1}{n}\right) - f(x) = -\gamma x^{-\gamma-1}.$$

But this difference equation also admits as a solution the expression

$$n^{1+\gamma} B_{-\gamma}(nx).$$

We have consequently

$$\sum_{s=0}^{n-1} B_{-\gamma}\left(x + \frac{s}{n}\right) = n^{1+\gamma} B_{-\gamma}(nx) + k \qquad \ldots \quad (6.1)$$

where k is a constant. To determine this constant, we have simply to differentiate both the sides of (6.1).

* *Vide* Whittaker and Watson *Modern Analysis* (13.14).

Observing that $D_x B_{-\gamma}(x) = -\gamma B_{-\gamma-1}(x)$ we infer that k ought to be zero. Therefore we get

$$\sum_{s=0}^{n-1} B_{-\gamma}\left(x + \frac{s}{n}\right) = n^{1+\gamma} B_{-\gamma}(nx) \qquad \ldots \quad (6.2)$$

From (4.5) and (6.2) we have the addition formula for the generalised zeta function

$$\sum_{s=1}^{n=1} \zeta\left(\gamma, x + \frac{s}{n}\right) = n^{1+\gamma}\,\zeta(\gamma, nx) \qquad \ldots \quad (6.3)$$

When $x = \frac{1}{n}$ we have

$$\sum_{s=0}^{n-1} \zeta\left(\gamma, \frac{s+1}{n}\right) = n^{1+\gamma}\,\zeta(\gamma). \qquad \ldots \quad (6.4)$$

TWO PROBLEMS IN THE THEORY OF LATTICE POINTS.

BY S. CHOWLA.

Introduction.—Let

$$Q(u) = \sum_{j=1}^{k} u_j^2 \qquad \ldots \quad (1)$$

and let $A(x) = A_Q(x)$ be the number of lattice points inside and on the ellipsoid $Q(u) \leqslant x$. The 'volume' of this ellipsoid is

$$V(x) = V_Q(x) = \frac{\pi^{k/2}\, x^{k/2}}{\Gamma(1 + k/2)}; \qquad \ldots \quad (2)$$

we write

$$P(x) = P_Q(x) = A_Q(x) - V_Q(x). \qquad \ldots \quad (3)$$

In what follows the letters Q, V and P have a similar significance for forms other than (1).

The order of $P(x)$ has been the subject of many investigations during the last ten years. Thus, we have

$P(x) =$	$O(x^{27/82})$	$(k = 2)$	(Nieland **1**)
$P(x) =$	$\Omega(x^{\frac{1}{4}} \log^{\frac{1}{4}} x)$	$(k = 2)$	(Hardy **1**)
$P(x) =$	$O(x^{43/58 + \varepsilon})$	$(k = 3)$	(Walfisz **2**)
$P(x) =$	$\Omega(x^{\frac{1}{2}} \log^{\frac{1}{2}} x)$	$(k = 3)$	(Szegö **1**)
$P(x) =$	$O\left(\dfrac{x \log x}{\log \log x}\right)$	$(k = 4)$	(Walfisz **1**)
$P(x) =$	$\Omega(x \log \log x)$	$(k = 4)$	(Walfisz **1**)
$P(x) =$	$O(x^{k/2-1})$	$(k \geqslant 5)$	(Landau **1**)
$P(x) =$	$\Omega(x^{k/2-1})$	$(k \geqslant 5)$	(Jarnik in Landau **2**)

The last two formulae show that the order of $P(x)$ for forms (1) has been exactly determined when $k \geqslant 5$. It is in this connection that

Jarnik (Jarnik 1) recently said that "Es wirkte daher höchst überraschend als es 1924 Herrn Walfisz gelang, das O-Problem für rationale Q mit $r \geqslant 8$ vollstandig zu losen."

My first object in this paper is to show that we can obtain exact results with certain other forms as well. Let

$$Q(u) = \sum_{j=1}^{6} u_j^2 + \sum_{j=7}^{k} u_j^4 \quad (k \geqslant 7). \quad \ldots \quad (4)$$

I show for this form that

$$P(x) = O(x^{\frac{k+2}{4}}) \quad (k \geqslant 7), \quad \ldots \quad (5)$$

$$P(x) = \Omega(x^{\frac{k+2}{4}}) \quad (k \geqslant 7). \quad \ldots \quad (6)$$

The question whether there is a k_0 such that we obtain exact results for the form

$$Q(u) = \sum_{j=1}^{k} u_j^4 \quad (k \geqslant k_0). \quad \ldots \quad (7)$$

I have not been able to answer.

The second problem I deal with is the following.

Let

$$P(x) = \sum_{n_1^4 + n_2^4 \leqslant x} 1 - \iint_{u_1^4 + u_2^4 \leqslant x} du_1\, du_2. \quad \ldots \quad (8)$$

Cauer, van der Corput and Jarnik (see Jarnik 2) have proved that

$$P(x) = O(x^{\frac{1}{6}}). \quad \ldots \quad (9)$$

Beyond (9) nothing is known about the behaviour of $P(x)$. It is not even known, for example, whether

$$P(x) = \Omega(\log\log\log\log\log\log x) \quad \ldots \quad (10)$$

or that

$P(n) < 0$, for infinitely many positive integers n. ... (11)

I prove that

$$\sum_{x \leqslant R} \sum_{n \leqslant x} P(n) = o(R^2). \quad \ldots \quad (12)$$

This result shows, at least, that for any positive ε, $P(n) < \varepsilon$, for infinitely many integers n. ... (13)

1. Proof of (5).

Lemma 1.

$$\sum_{x \leqq \sqrt[4]{R}} (R-x^4)^5 = AR^{21/4} - \tfrac{1}{2} R^5 + O(R^{5-5/4}) \qquad \text{... (14)}$$

Proof.—This result can be derived from Euler's summation formula. I give below a direct proof. Let $[x]$ denote the greatest integer contained in x, and $\theta = \sqrt[4]{R} - [\sqrt[4]{R}]$.

We have

$$\sum_{1}^{\sqrt[4]{R}} (R - x^4)^5 = \sum_{1}^{\sqrt[4]{R}-\theta} (R - x^4)^5$$

$$= \sum_{1}^{\sqrt[4]{R}-\theta} (R^5 - 5R^4 x^4 + 10R^3 x^8 - 10R^2 x^{12} + 5R x^{16} - x^{20})$$

$$= R^5(\sqrt[4]{R} - \theta) \qquad \text{... (15)}$$

$$-5R^4 \left(\frac{(\sqrt[4]{R}-\theta)^5}{5} + \tfrac{1}{2}(\sqrt[4]{R}-\theta)^4 + \frac{4}{6.2}(\sqrt[4]{R}-\theta)^3 - \frac{4.3.2}{30.24}(\sqrt[4]{R}-\theta)\right) \qquad \text{... (16)}$$

$$+10R^3 \left(\frac{(\sqrt[4]{R}-\theta)^9}{9} + \tfrac{1}{2}(\sqrt[4]{R}-\theta)^8 + \frac{8}{6.2}(\sqrt[4]{R}-\theta)^7 - \frac{8.7.6}{30.24}(\sqrt[4]{R}-\theta)^5 + O(R^{\frac{3}{4}})\right) \qquad \text{... (17)}$$

$$-10R^2 \left(\frac{(\sqrt[4]{R}-\theta)^{13}}{13} + \tfrac{1}{2}(\sqrt[4]{R}-\theta)^{12} + \frac{12}{6.2}(\sqrt[4]{R}-\theta)^{11} - \frac{12.11.10}{30.24}(\sqrt[4]{R}-\theta)^9 + O(R^{\frac{7}{4}})\right) \qquad \text{... (18)}$$

$$-5R \left(\frac{(\sqrt[4]{R}-\theta)^{17}}{17} + \tfrac{1}{2}(\sqrt[4]{R}-\theta)^{16} + \frac{16}{6.2}(\sqrt[4]{R}-\theta)^{15} - \frac{16.15.14}{30.24}(\sqrt[4]{R}-\theta)^{13} + O(R^{\frac{11}{4}})\right) \qquad (19)$$

$$- \left(\frac{(\sqrt[4]{R}-\theta)^{21}}{21} + \tfrac{1}{2} (\sqrt[4]{R}-\theta)^{20} \right.$$

$$\left. + \frac{20}{6.2} (\sqrt[4]{R}-\theta)^{19} - \frac{20.19.18}{30.24} (\sqrt[4]{R}-\theta)^{17} + O(R^{\frac{15}{4}}) \right) \quad \ldots (20)$$

The co-efficient of $R^{5+1/4}$ collected from the expressions (15) to (20) is

$$\int_0^1 (1-x^4)^5 \, dx. \qquad \ldots \quad (21)$$

The co-efficient of R^5 in the same expressions is

$$-\theta - 5(-\theta+\tfrac{1}{2}) + 10(-\theta+\tfrac{1}{2}) - 10(-\theta+\tfrac{1}{2})$$
$$+ 5(-\theta+\tfrac{1}{2}) - (-\theta+\tfrac{1}{2}) = -\tfrac{1}{2}.$$

We shall now collect the co-efficient of $R^{5-1/4}$ in the same expressions, and we shall see that it is 0.

This co-efficient is

$$-5\left(\frac{5.4\,\theta^2}{5.1.2} - \tfrac{1}{2}.4\theta + \frac{4}{12}\right) + 10\left(\frac{9.8\,\theta^2}{9.1.2} - \tfrac{1}{2}.8\theta + \frac{8}{12}\right)$$

$$-10\left(\frac{13.12\theta^2}{13.1.2} - \tfrac{1}{2}.12\,\theta + \frac{12}{12}\right) + 5\left(\frac{17.16}{17.1.2}\theta^2 - \tfrac{1}{2}.16\theta + \frac{16}{12}\right)$$

$$-1\left(\frac{21.20}{21.1.2}\theta^2 - \tfrac{1}{2}.20\,\theta + \frac{20}{12}\right) \qquad \ldots \quad (22)$$

$$= A\theta^2 \left(\frac{d}{dx}(1-x^4)^5\right)_{x=1} + B\theta\left(\frac{d}{dx}(1-x^4)^5\right)_{x=1}$$
$$+ C\left(\frac{d}{dx}(1-x^4)^5\right)_{x=1} = 0. \qquad \ldots \quad (23)$$

Similarly it can be shown that the co-efficients of $R^{5-m/4}$ ($m = 1, 2, 3, 4$) vanish; because

$$\left(\frac{d^m}{dx^m}(1-x^4)^5\right)_{x=1} = 0 \qquad (m = 1, 2, 3, 4). \quad \ldots \quad (24)$$

The lemma is now proved. We note that our method shows the truth of the following more general result:

Lemma 2. If $m \equiv 0 \pmod{k}$, *then*

$$\sum_{x \leqslant R^{1/k}} (R-x^k)^{m/k} = AR^{(m+1)/k} - \tfrac{1}{2} R^{m/k} + O(R^{m/k-m/k^2})$$
$$(m > 0). \quad \ldots \quad (25)$$

[In what follows numerals in thick print (when not accompanied by a name) are te be regarded as references to the equations of this paper.]

Let $A_{10}(x)$, $P_{10}(x)$, $V_{10}(x)$ relate **(1, 2, 3)** to the form

$$\sum_{j=1}^{10} u_j^2 = x, \qquad \dots \quad (26)$$

and let $A_{11}(x)$, $P_{11}(x)$, $V_{11}(x)$ relate to the form

$$\sum_{j=1}^{10} u_j^2 + u_{11}^4 = x. \qquad \dots \quad (27)$$

Further, let

$$V_{10}(x) = C_{10}\, x^5,\ V_{11}(x) = C_{11}\, x^{\frac{21}{4}}. \qquad \dots \quad (28)$$

We have

$$A_{11}(x) = V_{11}(x) + P_{11}(x) = C_{11}\, x^{21/4} + P_{11}(x)\ (\mathbf{3, 28}) \qquad \dots \quad (29)$$

$$= A_{10}(x) + 2 \sum_{1}^{\sqrt[4]{x}} A_{10}(x - n^4)$$

$$= P_{10}(x) + V_{10}(x) + 2 \sum_{1}^{\sqrt[4]{x}} C_{10}(x - n^4)^5 + 2 \sum_{1}^{\sqrt[4]{x}} P_{10}(x - n^4)$$

$$= P_{10}(x) + C_{10}\, x^5 + (C_{11}\, x^{51/4} - C_{10}\, x^5 + O(x^{5-5/4}))$$

$$+ 2 \sum_{1}^{\sqrt[4]{x}} P_{10}(x - n^4)\ (\mathbf{14}). \qquad \dots \quad (30)$$

By Landau (Landau **1**),

$$P_{10}(x) = O(x^4). \qquad \dots \quad (31)$$

Hence

$$P_{11}(x) = O(x^{4\frac{1}{4}})\ (\mathbf{29,\ 30,\ 31}). \qquad \dots \quad (32)$$

In the rest of this section $A_k^*(x)$, $V_k^*(x)$, $P_k^*(x)$ relate to the form

$$Q_k^*(u) = \sum_{j=1}^{6} u_j^2 + \sum_{j=7}^{k} u_j^4 = x \quad (k \geqslant 7). \qquad \dots \quad (33)$$

The method used to prove (32) shows easily that (we apply (25) with $k = 4$, $m = 12$)

$$P_7^*(x) = O(x^{2\frac{1}{4}}). \qquad \dots \quad (34)$$

Hence (5) is true with $k = 7$. From (34) we shall deduce (5) by induction. Let, then

$$P_k^*(x) = O(x^{\frac{k+2}{4}}) \quad (k \geqslant 7). \qquad \dots \quad (35)$$

From (5) and (35) we shall deduce that

$$P_{k+1}^*(x) = O(x^{\frac{k+3}{4}}) \qquad (k \geqslant 7). \qquad \dots \quad (36)$$

We need the following

Lemma 3. If $m \geqslant \frac{13}{4}$, *then*

$$\sum_{x \leqslant \sqrt[4]{R}} (R - x^4)^m = AR^{m+1/4} - \tfrac{1}{2} R^m + O(R^{m-3/4}). \quad \dots \quad (37)$$

Proof.—We have, for every integer $k \geqslant 0$ (see K. Knopp "*Theory and Application of Infinite series*," translated by R. C. Young, p. 526), provided only that the derivatives of $f(x)$ involved exist and are continuous,

$$f_0 + f_1 + \dots + f_n = \int_0^n f(x)\,dx + \tfrac{1}{2}(f_0 + f_n) + \frac{B_2}{2!}(f_n' - f_0') \quad \dots \quad (38)$$

$$+ \frac{B_4}{4!}(f_n''' - f_0''') + \dots + \frac{B_{2k}}{(2k)!}\left(f_n^{(2k-1)} - f_0^{(2k-1)}\right)$$

$$+ \int_0^n P_{2k+1}(x) f^{(2k+1)}(x)\,dx,$$

where

$$P_{2\lambda}(x) = (-1)^{\lambda-1} \sum_{n=1}^{\infty} \frac{2 \cos 2n\pi x}{(2n\pi)^{2\lambda}}, \quad \dots \quad (39)$$

$$P_{2n+1}(x) = (-1)^{\lambda-1} \sum_{n=1}^{\infty} \frac{2 \sin 2n\pi x}{(2n\pi)^{2\lambda+1}}, \quad \dots \quad (40)$$

$$P_{2\lambda}(0) = (-1)^{\lambda-1} \sum_{n-1}^{\infty} \frac{2}{(2n\pi)^{2\lambda}} = \frac{B_{2\lambda}}{(2\lambda)!}. \quad \dots \quad (41)$$

We apply (38) with

$$f(x) = (R - x^4)^m, \; m \geqslant \frac{13}{4}, \; n = [\sqrt[4]{R}], \; k = 1. \quad \dots \quad (42)$$

Then

$$\frac{f'(x)}{m} = -4x^3(R-x^4)^{m-1} \qquad \dots \quad (43)$$

$$\frac{f''(x)}{m} = -12x^2(R-x^4)^{m-1} + 16x^6(m-1)(R-x^4)^{m-2} \dots \quad (44)$$

$$\frac{f'''(x)}{m} = -24x(R-x^4)^{m-1} + 48x^5(m-1)(R-x^4)^{m-2} \dots \quad (45)$$

$$+ 96x^5(m-1)(R-x^4)^{m-2} - 64x^9(m-1)(m-2)(R-x^4)^{m-2}.$$

Thus, if $0 \leqslant x \leqslant \sqrt[4]{R}$,

$$f'''(x) = O(R^{m-3/4}) \quad (45). \qquad \dots \quad (46)$$

Hence, by splitting up the range $(0, n)$ into parts in each of which $f'''(x)$ is monotone, and applying the second mean-value theorem in each, we obtain

$$\int_0^n P_3(x) f'''(x)\, dx = O(R^{m-3/4}). \qquad \dots \quad (47)$$

Further

$$f_n = O(R^{\frac{3}{4}m}) = O(R^{m-\frac{3}{4}});\ f_n' = O(R^{\frac{3}{4}m}) = O(R^{m-\frac{3}{4}}). \dots \quad (48)$$

Hence

$$\sum_{x \leqslant \sqrt[4]{R}} (R-x^4)^m = AR^{m+\frac{1}{4}} - \tfrac{1}{2}R^m + O(R^{m-\frac{3}{4}}) \qquad \dots \quad (49)$$

(38, 47, 48)

the desired result.

We now return to the proof of (36). We have:

$$P_{k+1}^*(x) = A_{k+1}^*(x) - V_{k+1}^*(x) = A_k^*(x) + 2\sum_{n \leqslant \sqrt[4]{x}} A_k^*(x-n^4) - V_{k+1}^*(x)$$

$$= P_k^*(x) + V_k^*(x) + 2\sum_{n \leqslant \sqrt[4]{x}} V_k^*(x-n^4) + 2\sum_{n \leqslant \sqrt[4]{x}} P_k^*(x-n^4) - C_{k+1}x^{(k+7)/4}$$

$$= O(x^{(k+2)/4}) + C_k x^{3+(k-6)/4} + \{C_{k+1}x^{(k+7)/4} - C_k x^{(k+6)/4}$$

$$+ O(x^{(k+3)/4})\} + O(x^{(k+3)/4}) - C_{k+1}x^{(k+7)/4} \quad \textbf{(49, 35)}$$

$$= O(x^{(k+3)/4}), \qquad \dots \quad (50)$$

and thus (5) has been proved by induction (**34, 35, 50**).

2. Proof of (6).

Let n be an integer > 0; from the obvious relation

$$A_k^* (n + \tfrac{1}{4}) = A_k^* (n + \tfrac{1}{2}),$$

it follows that

$$P_k^* (n + \tfrac{1}{4}) - P_k^* (n + \tfrac{1}{2}) = V_k^* (n + \tfrac{1}{2}) - V_k^* (n + \tfrac{1}{4}) \quad (3)$$

$$= C_k \{ (n + \tfrac{1}{2})^{3 + \frac{k-6}{4}} - (n + \tfrac{1}{4})^{3 + \frac{k-6}{4}} \}$$

$$= \tfrac{1}{4} C_k \frac{k+6}{4} n^{2 + \frac{k-6}{4}} + O(n^{1 + \frac{k-6}{4}})$$

whence

$$P_k^* (x) = \Omega (x^{\frac{k+2}{4}})$$

which is (6).

3. Proof of (12).

In this section R (x) denotes the fractional part of x. Let A (x), P (x), V (x) relate (1, 2, 3) to the form $u_1^4 + u_2^4 = x$. We write $c_n = 1$ if $n = k^4$ (k, an integer) $c_n = 0$ otherwise.

Lemma 4. If x is an integer,

$$\sum_{n \leqslant x} R(\sqrt[4]{n}) = \tfrac{1}{2} x + x^{\frac{3}{4}} (2\theta^2 - 2\theta + \tfrac{1}{3})$$

$$+ \sqrt{x} (-2\theta^3 + 3\theta^2 - \theta) + x^{\frac{1}{4}} (\theta^4 - 2\theta^3 + \theta^2 - \tfrac{8}{15}) + O(1). \quad \ldots (51)$$

where

$$\theta = R(\sqrt[4]{x}).$$

Proof.—Behnke (Behnke 1) has proved that the numbers R $(\sqrt[4]{n})$ $(n = 1, 2, 3, \ldots)$ are "gleichverteilt" and hence by Weyl (Weyl 1),

$$\sum_{n \leqslant x} R(\sqrt[4]{n}) \sim \tfrac{1}{2} x. \quad \ldots (52)$$

This result is not as sharp as (51), which we now prove. We have

$$\sum_{n \leqslant y} C_n = \sqrt[4]{y} - R(\sqrt[4]{y}), \quad \ldots (53)$$

$$\sum_{y \leqslant x} (\sqrt[4]{y} - R(\sqrt[4]{y})) = \sum_{n \leqslant x} C_n (x + 1 - n) \quad (53)$$

$$= (x + 1)(\sqrt[4]{x} - R(\sqrt[4]{x})) - \sum_{1}^{\sqrt[4]{x} - R(\sqrt[4]{x})} n^4$$

$$= (x+1)(\sqrt[4]{x} - \theta - \left(\frac{(\sqrt[4]{x}-\theta)^5}{5} + \frac{1}{2}(\sqrt[4]{x}-\theta)^4\right.$$

$$\left. + \frac{4}{12}(\sqrt[4]{x}-\theta)^3 - \frac{24}{30.24}(\sqrt[4]{x}-\theta)\right) \quad \ldots \quad (54)$$

$$\sum_{y \leqslant x} \sqrt[4]{y} = \frac{4}{5} x^{\frac{5}{4}} + \frac{1}{2} x^{\frac{1}{4}} + O(1). \quad \ldots \quad (55)$$

(51) follows from (54) and (55).

Lemma 5.

$$\sum_{x \leqslant \sqrt[4]{R}} (R - x^4)^{\frac{5}{4}} = AR^{\frac{5}{2}} - \tfrac{1}{2} R^{\frac{3}{4}} + O(R^{\frac{15}{16}}). \quad \ldots \quad (56)$$

Proof.—If the derivates of $f(x)$ involved exist and are continuous, we have (see K. Knopp, *Ibid*, pp. 524 and 525)

$$f_0 + f_1 + \ldots + f_n = \int_0^n f(x)\,dx + \tfrac{1}{2}(f_0 + f_n) + \int_0^n P_1(x) f'(x)\,dx$$

$$= \int_0^n f(x)\,dx + \frac{1}{2}(f_0 + f_n) + \frac{1}{12}(f'_n - f'_0)$$

$$- \int_0^n P_2(x) f''(x)\,dx \ (39, 40) \quad \ldots \quad (57)$$

We apply (57) with

$$f(x) = (R - x^4)^{\frac{5}{4}},\ n = [\sqrt[4]{R}] - 2. \quad \ldots \quad (58)$$

Then

$$f'(x) = -5x^3(R - x^4)^{\frac{1}{4}}, \quad \ldots \quad (59)$$

$$f''(x) = -15x^2(R - x^4)^{\frac{1}{4}} + 5x^6(R - x^4)^{-\frac{3}{4}}. \quad \ldots \quad (60)$$

Hence, for $0 \leqslant x \leqslant [\sqrt[4]{R}] - 2$,

$$f''(x) = O(R^{\frac{15}{16}}). \quad \ldots \quad (61)$$

Further,

$$f(n) = O(R^{\frac{15}{16}}),\ f'(n) = O(R^{\frac{15}{16}}), \quad \ldots \quad (62)$$

$$\int_0^n P_2(x) f''(x)\,dx = O(R^{\frac{15}{16}}) \quad (61). \quad \ldots \quad (63)$$

(56) follows from (57), (62), (63).

We are now in a position to prove (12).

We have,

$$A(x) = 1+4 \sum_{n \leqslant \sqrt[4]{x}} \left[\sqrt[4]{x-n^4}\right] + 4[\sqrt[4]{x}], \qquad \dots \quad (64)$$

$$P(x) + V(x) = 1 + 4 \sum_{n \leqslant \sqrt[4]{x}} \sqrt[4]{x-n^4} - 4 \sum_{n \leqslant \sqrt[4]{x}} R(\sqrt[4]{x-n^4})$$

$$+ 4\sqrt[4]{x} - 4R(\sqrt[4]{x})$$

$$= 1+4 \sum_{n \leqslant x} c_n \sqrt[4]{x-n} - 4 \sum_{n \leqslant x} c_n R(\sqrt[4]{x-n}) + 4\sqrt[4]{x} - 4R(\sqrt[4]{x})$$

$$\dots \quad (65)$$

$$\sum_{n \leqslant x} (P(n) + v(n)) = x + 4 \sum_{n \leqslant x} c_n \sum_{1}^{x-n} \sqrt[4]{r}$$

$$-4 \sum_{n \leqslant x} c_n \sum_{1}^{x-n} R(\sqrt[4]{r}) + \frac{16}{5} x^{\frac{5}{4}} - 2x + O(x^{\frac{3}{4}}) \quad (51)$$

$$= x + 4 \sum_{n \leqslant x} c_n \left(\frac{4}{5}(x-n)^{\frac{5}{4}} + O(x^{\frac{1}{4}})\right)$$

$$-4 \sum_{n \leqslant x} c_n \left\{ \tfrac{1}{2}(x-n) + (x-n)^{\frac{3}{4}} (2R^2(\sqrt[4]{x-n}) - 2R(\sqrt[4]{x-n}) + \tfrac{1}{3}) \right\}$$

$$+ \frac{16}{5} x^{\frac{5}{4}} - 2x + O(x^{\frac{3}{4}}) \quad (51)$$

$$= x + Ax^{\frac{3}{2}} - \frac{8}{5} x^{\frac{5}{4}} + O(x^{\frac{15}{16}}) - 2 \sum_{n \leqslant x} c_n (x-n) \qquad \dots \quad (66)$$

$$-4 \sum_{n \leqslant x} c_n (x-n)^{\frac{3}{4}} (2\theta^2 - 2\theta + \tfrac{1}{3}) + \frac{16}{5} x^{\frac{5}{4}} - 2x + O(x^{\frac{3}{4}})$$

$$(\theta = R(\sqrt[4]{x-n})). \quad (56)$$

$$\sum_{n \leqslant x} c_n (x-n) = x \sum_{1}^{\sqrt[4]{x}} 1 - \sum_{1}^{\sqrt[4]{x}} n^4 = \frac{4}{5} x^{\frac{5}{4}} - \frac{1}{2} x + O(x^{\frac{3}{4}}) \qquad \dots \quad (67)$$

$$\sum_{n \leqslant x} (P(n) + V(n)) = Ax^{\frac{3}{2}} - 4 \sum_{n \leqslant x} c_n (x - n^{\frac{3}{4}})$$

$$(2R^2 (\sqrt[4]{x - n}) - 2R (\sqrt[4]{x - n}) + \tfrac{1}{3}$$

$$+ O(x^{15/16}) \quad \mathbf{(66,\ 67)}. \qquad \ldots \quad (68)$$

$$\sum_{n \leqslant x} P(n) = -4 \sum_{n \leqslant x} c_n (x - n)^{\frac{3}{4}} (2R^2 (\sqrt[4]{x - n}) - 2R (\sqrt[4]{x - n}) + \tfrac{1}{3})$$

$$+ O(x)^{15/16}. \quad \mathbf{(68)} \qquad \ldots \quad (69)$$

$$\sum_{x \leqslant R} \sum_{n \leqslant x} P(n) = O(R^{31/16})$$

$$-4 \sum_{n \leqslant R} c_n \left\{ \sum_{1}^{R-n} m^{\frac{3}{4}} (2R^2 (\sqrt[4]{m}) - 2R (\sqrt[4]{m}) + \tfrac{1}{3}) \right\} \quad \mathbf{(69)}$$

$$\ldots \quad (70)$$

Now the nnmbers $R(\sqrt[4]{m})$ $(m = 1, 2, 3, \ldots)$ are "gleichverteilt" (Behnke **1**). Hence by Weyl (Weyl **1**)

$$\sum_{1}^{x} (2R^2) \sqrt[4]{n}) - 2R (\sqrt[4]{n}) + \tfrac{1}{3}) = 0\,(x)$$

and hence by partial summation

$$\sum_{1}^{x} n^{\frac{3}{4}} (2R^2 (\sqrt[4]{n}) - 2R (\sqrt[4]{n}) + \tfrac{1}{3}) = 0\,(x^{\frac{7}{4}}).$$

From (70) and (71),

$$\sum_{x \leqslant R} \sum_{n \leqslant x} P(n) = 0\,(R^2), \qquad \ldots \quad (72)$$

which is (12).

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ON SYSTEMS OF CURVES ON A RING-SHAPED SURFACE

By Dr. A. Weil.

Poincaré, in his paper of 1885 entitled "Sur les courbes définies par les équations différentielles"* has devoted a whole chapter (pp. 137—158 of the *Oeuvres*) to the study, under certain restrictive assumptions, of the systems of curves defined on a ring-shaped surface by a differential equation of the first order. He makes use of the "method of consequents," which he has also successfully applied to other problems. I shall give here a proof of Poincaré's results by a new method, which may have some advantages over that of Poincaré in this case, and possibly in other problems.

1. I shall follow Poincaré in replacing the original problem by the study of the solutions of a differential equation:

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y)$ is analytic and periodic both in x and in y with the period 1†.

By a characteristic we shall understand any curve $y = \phi(x)$ which is a solution of the differential equation. From the periodicity of f it follows that if $y = \phi(x)$ is a characteristic, and if p, q are any integers, the curve $y - q = \phi(x - p)$, deduced from $y = \phi(x)$ by a translation (p, q), is also a characteristic.

A point (ξ, η) is said to be above or below a characteristic $y = \phi(x)$ according as $\eta - \phi(\xi)$ is > 0 or < 0. Since no two characteristics can have a point in common unless they coincide, out of two distinct characteristics one must be entirely above the other.

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* *J. de Liouville*, (IV) t. 1, p. 167; and *Oeuvres*, t. 1, pp. 90—158.

† Instead of the analyticity, it is enough to assume a Cauchy-Lipschitz condition for $f(x, y)$.

2. We shall now choose one particular characteristic C, given by $y = \phi(x)$; and by C (p, q) we shall understand the characteristic $y = q + \phi(x - p)$ deduced from C by the translation (p, q), p and q being any two integers. C (0, 0) is the same as C itself.

It is clear that C (0, 1) is above C, and C (0, —1) below C. Moreover, if C (p, q) is above C, C $(2p, 2q)$ will be above C (p, q) C $(3p, 3q)$ above C $(2p, 2q)$, etc., so that, when k is any positive integer, C (kp, kq) is above C. Similarly, C $(-p, -q)$, and therefore also C $(-kp, -kq)$, will be below C.

Now we shall put any rational number $\frac{q}{p}$ (p and q being integers, and $p > 0$) into class L or into class R, according as C (p, q) is below C or above C: from the preceding paragraph it follows that if $\frac{q}{p} = \frac{q'}{p'}$, $p > 0$, $p' > 0$, then C (p, q) and C (p', q') are both below C, both above C, or both coincide with C, so that there can be no ambiguity in the classification. The two classes exist, because if q is positive and very large C $(1, q)$ is above C and C $(1, -q)$ below C. If $\frac{q}{p}$ is not in class L, and if $\frac{s}{r} > \frac{q}{p}$, then $\frac{s}{r}$ is in R: C (rp, rq) is namely either above C or coincident with C; C (rp, sp) is above C (rp, rq) since it is deduced from it by the translation $(0, sp - rq)$, and $sp - rq > 0$; therefore C (rp, sp) is above C, and $\frac{sp}{rp} = \frac{s}{r}$ is in R. Similarly, if $\frac{q}{p}$ is not in R and if $\frac{s}{r} < \frac{q}{p}$, then $\frac{s}{r}$ is in L. This shows that all rational numbers, with possibly one exception, are included either in L or in R, and that L and R define a number μ, rational or irrational.

If $\frac{n}{m}$ is neither in L nor in R, then $\mu = \frac{n}{m}$; C (m, n) coincides with C; $\phi(x) - \frac{n}{m}x$ is a periodic function of x with the period m. We shall henceforward suppose that such is not the case.

3. p being given, let q be the greatest integer such that q/p is in L: then $(q+1)/p$ is in R, and $q \leqslant \mu p \leqslant q+1$. Any point $\{x+p, \phi(x)+q\}$

of C (p, q) is below C, and any point $\{ x+p, \phi(x)+q+1 \}$ of C $(p. q+1)$ is above C ; in other words

$$\phi(x)+q < \phi(x+p) < \phi(x) + q + 1.$$

X being now any number > 0, let us apply these inequalities to $p = [X]$ and $x = (X)$ (integral and fractional parts of X). We have $p \leqslant x < p+1$, and if α and β are the minimum and maximum values of $\phi(x)$ in the interval (0, 1), we have $\alpha \leqslant \phi(x) \leqslant \beta$. Hence we ultimately obtain the inequality

$$\mu X + (\alpha - \mu - 1) < \phi(X) < \mu X + (\beta + 1).$$

The same again holds good for negative values of X, as we find by applying the above inequality to $x = X$ and $p = -[X]$. Thus we can write

$$\phi(X) = \mu X + 0(1).$$

This shows that the curve C has an asymptotic direction parallel to $y = \mu x$, and that it remains within a strip of finite breadth paralle to that direction.

4. To every characteristic C (p, q) we now attach the number $\tau = q - \mu p$, which we call its parameter ; and by the parameter τ of any other characteristic D, we shall understand the upper bound of the parameters of all curves C (p, q) which are below D. If C (p, q) is below D, C $(p+r, q+s)$ is below the curve D (r, s) deduced from D by the translation (r, s) : hence the parameter of D (r, s) is $\tau + s - \mu r$. We now define the function $\tau(x, y)$ as the parameter of the characteristic going through the point (x, y). When x is kept constant, $\tau(x, y)$ is obviously a non-decreasing function of y ; and we have

$$\tau(x+r, y+s) = \tau(x, y) + s - \mu r.$$

If μ is rational and $= n/m$, with n prime to m, $\tau(x, y)$ can take all values of the form k/m (k being any integer) and only those, and so is a discontinuous function. In that case, x_0 being arbitrarily chosen, let y_0 be the upper bound of the values of y for which $\tau(x_0, y) \leqslant 0$; and let D be the characteristic going through (x_0, y_0) ; any characteristic below D has a parameter $\leqslant 0$, and any characteristic above D has a para-

meter > 0: and the same will be true of the curves D $(\pm m, \pm n)$ deduced from D by the translations $(\pm m, \pm n)$, since these translations do not alter the parameter of any characteristic. Now, either these two curves coincide with D; or else one of them, say D (m, n), will be below D: and then there will be some characteristic E below D and above D (m, n), the parameter of which will be at the same time $\leqslant 0$ and >0. This is a contradiction; so we conclude that D is transformed into itself by the translation (m, n); hence its equation is of the form $y = \psi(x)$ where $\psi(x) - n/m \cdot x$ is a periodic function of x with the period m.

5. Leaving to the reader a more detailed study of this case, we shall now assume that μ is irrational; so that in any interval (β, γ) there are infinitely many numbers of the form $q - \mu p$. Then $\tau(x, y)$ is a continuous function of x, y: if namely $\tau(a, b) = \alpha$, and if ε is any positive quantity, we can find p, q and p', q' such that:

$$\alpha - \varepsilon < q - \mu p < \alpha < q' - \mu p' < \alpha + \varepsilon.$$

Then the curve C (p, q) will be below the point (a, b), C (p', q') will be above it; and any point between these two curves, and in particular any point within a sufficiently small neighbourhood of (a, b) will have a parameter comprised between $q - \mu p$ and $q' - \mu p'$.

We have now to distinguish between two cases: either $\tau(x, y)$ considered as a function of y alone (x being kept constant) is always increasing, or else it is constant in some intervals. This does not depend upon the particular value given to x: if, for instance for $x = x_0$, τ has the constant value τ_0 within the interval (y_0, y_1), it will have the same value at every point comprised between the two characteristics D_0 and D_1 which go through (x_0, y_0) and (x_0, y_1) respectively, and so, on any line $x =$ constant there will be an interval within which τ has the same constant value. Such values τ_0, taken by τ on infinitely many characteristics, will be called *critical values* of $\tau(x, y)$; and the region comprised between the lowest and the highest characteristic with the parameter τ_0 will be called the *critical strip* corresponding to that value. If $\tau(x, y)$ has a critical value τ_0, all numbers $\tau_0 + q - \mu p$ will be critical values of τ, and so there will be infinitely many critical values in every interval (and infinitely many critical strips between any two characteristics).

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Notes and Questions.

On a Problem in the Geometry of the Triangle.

Introduction.—The problem is to find a point D in the plane of a triangle ABC such that circles can be inscribed in each of the three proper quadrilaterals formed by the joins of the four points That is, if P, Q, R be the points of intersection of (AD, BC), (AC, BD), (AB, CD), the problem is to determine the configuration of A, B, C, D so that it may be possible to inscribe circles in the quadrilaterals ARDQ, BPDR, CQDP. The solution of the analogous problem in which the three quadrilaterals are required to be cyclic is known; for then D is the ortho-centre of ABC and is the only point satisfying the conditions. A similar problem is that of finding a point such that if perpendiculars be drawn to the sides, circles can be inscribed in each of the three resulting quadrilaterals.* The point is the in-centre or one of the ex-centres.

The problem stated in the paper has not, as far as the writer is aware, been dealt with before. It is reminiscent of Malfatti's† problem of describing three circles mutually in contact, each of which touches two sides of a given triangle. Malfatti's problem has eight solutions. The problem in this paper is however not based on, or otherwise connected with, Malfatti's problem.

1. *Position of D.*

Let a circle of radius ρ_1 touch AB, AC and let BDQ, CDR be the other tangents from B, C intersecting in D. Writing M (ρ) for the length of the tangent from M to the circle of radius ρ, we get

$$\begin{aligned} AB + CD &= A(\rho_1) + B(\rho_1) + C(\rho_1) - D(\rho_1) \qquad \dots (1) \\ &= AC + BD. \end{aligned}$$

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* Todhunter and Hogg: *Plane Trigonometry* (1911). p. 204, Ex. 73.

† *Vide* Casey's *Sequel to Euclid* for a geometrical solution due to Steiner and Hobson: *Plane Trigonometry* (1897). p. 209, for an analytical solution based on that of Lechmutz.

Hence we have to find a point D such that

$$AB + CD = AC + BD = BC + AD. \qquad \dots (2)$$

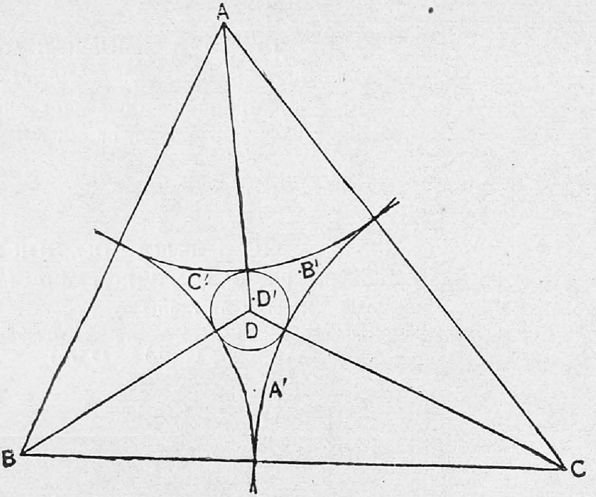
Writing $\quad BC = \beta + \gamma,\ CA = \gamma + \alpha,\ AB = \alpha + \beta, \qquad \dots (3)$

and $\quad AB + CD = K = \alpha + \beta + \gamma + \delta$

we get $\quad AD = \alpha + \delta,\ \ BD = \beta + \delta,\ \ CD = \gamma + \delta. \qquad \dots (4)$

Hence D is the centre of a circle of radius δ touching the circles of radii α, β, γ and having their centres at A, B, C respectively. Hence A, B, C, D are the centres of four circles which touch one another externally. It is obvious that the circles (A, α), (B, β), (C, γ), pass through the points of contact of the in-circle of the triangle with the sides and that the circle (D, δ) touches them all externally and is situated within the region formed by the smaller arcs of the first three circles. The number of points D satisfying the problem is discussed in § 5 below.

2. *Notation.* Following Mr. Gulasekharam we denote the curvatures of the four circles (A, α), (B, β), (C, γ), (D, δ) by c_1, c_2, c_3, c_4;



and the curvatures and centres of the in-circles of the triangles DBC, DCA, DAB, and ABC by d_1, d_2, d_3, $-d_4$ and A′, B′, C′, D′. The

following relations then hold good.*

$$\sum_{1}^{4} c_1^2 = 2 \sum_{1}^{6} c_1 c_2. \qquad \ldots (5)$$

$$\sum_{1}^{4} d_1^2 = 2 \sum d_1 d_2, \qquad \ldots (6)$$

$$d_1^2 = c_2 c_3 + c_3 c_4 + c_4 c_2, \quad \text{etc.} \qquad \ldots (7)$$

$$c_1^2 = d_2 d_3 + d_3 d_4 + d_4 d_2, \quad \text{etc.} \qquad \ldots (8)$$

$$c_1 + d_1 = c_2 + d_2 = c_3 + d_3 = c_4 + d_4$$
$$= \tfrac{1}{2}(c_1 + c_2 + c_3 + c_4)$$
$$= \tfrac{1}{2}(d_1 + d_2 + d_3 + d_4). \qquad \ldots (9)$$

3. *Relation between the mutual distances.*

(i) $$AB - AC = DB - DC.$$

Hence A, D lie on a hyperbola (say H_{23}) with foci at B, C. The centre of the ρ_1 circle is the pole of the chord AD of the hyperbola.†

(ii) $$BR + RC = B(\rho_1) - R(\rho_1) + R(\rho_1) + C(\rho_1)$$
$$= B(\rho_1) + C(\rho_1) = BQ + QC.$$

Hence Q, R lie on an ellipse (say E_{23}) with foci at B, C.

(iii) $$AR - RD = AQ - QD.$$

∴ Q, R lie on a hyperbola (say H'_{14}) whose foci are A, D.

(iv) $$RB - BQ = QC - RC.$$

∴ B, C lie on a hyperbola (say H_{qr}) whose foci are Q, R.

(v) Let ρ_1, ρ_2, ρ_3 be the radii of the three circles inscribed in the quadrilaterals ARDQ, BPDC, CQDP.

$$BP = AB + AP - 2A(\rho_2)$$
$$PC = AP + AC - 2A(\rho_3).$$

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* *Vide* this *Journal*, Vol. XVI, Part 2, p. 11.

† Casey: *Analytical Geometry* (1885), page 182, Ex. 14.

$$\therefore \tfrac{1}{2}\Sigma(BP - PC) = A(\rho_3) - A(\rho_2) + B(\rho_1) - B(\rho_3) + C(\rho_2) - C(\rho_1)$$

$$= B(\rho_1) - C(\rho_1) + C(\rho_2) - A(\rho_2) + A(\rho_3) - B(\rho_3)$$

$$= c - b + a - c + b - a = 0.$$

$$\therefore \quad BP + CQ + AR = PC + QA + RB. \quad \ldots \quad (10)$$

Similarly in the triangle DBC

$$BP + CR - DQ = PC - RD + QB. \quad \ldots \quad (11)$$

(vi) $$BR + RC - BC = 2.R(\rho_2)$$

$$= BQ + QC - BC$$

$$= 2Q(\rho_3).$$

$$\therefore \quad Q(\rho_3) = R(\rho_2);\ R(\rho_1) = P(\rho_3);\ P(\rho_2) = Q(\rho_1). \ldots \quad (12)$$

(vii) The areas of the triangles DBC, DCA, DAB, ABC are

$$\Delta_1 = c_1 d_1 (c_1 c_2 c_3 c_4)^{-1}$$

$$\Delta_2 = c_2 d_2 (c_1 c_2 c_3 c_4)^{-1}$$

$$\Delta_3 = c_3 d_3 (c_1 c_2 c_3 c_4)^{-1}$$

$$\Delta_4 = -c_4 d_4 (c_1 c_2 c_3 c_4)^{-1}. \quad \ldots \quad (13)$$

Hence we get the identical relation

$$c_1 d_1 + c_2 d_2 + c_3 d_3 + c_4 d_4 = 0. \quad \ldots \quad (14)$$

(viii) $$\cot \tfrac{1}{2} A = \sqrt{\frac{\alpha(\alpha+\beta+\gamma)}{\beta\gamma}} = \frac{-d_4}{c_1} \quad \ldots \quad (15)$$

4. *The circles of radii ρ_1, ρ_2, ρ_3 and tangents from the vertices.*

$$\tfrac{1}{2}\rho_1(AB + CD + BD + CA) = \Delta_2 + \Delta_3.$$

$$\therefore \quad \left.\begin{aligned} \rho_1 &= (c_2 d_2 + c_3 d_3)(\Sigma\, c_2 c_3 c_4)^{-1}; \\ \rho_2 &= (c_3 d_3 + c_1 d_1)(\Sigma\, c_2 c_3 c_4)^{-1}; \\ \rho_3 &= (c_1 d_1 + c_2 d_2)(\Sigma\, c_2 c_3 c_4)^{-1}. \end{aligned}\right\} \quad \ldots \quad (16)$$

In terms of α, β, γ, δ we get the relation

$$(\rho_1 + \rho_2 + \rho_3)(\alpha + \beta + \gamma + \delta) = 2\sqrt{\alpha\beta\gamma(\alpha+\beta+\gamma)}. \quad \ldots \quad (17)$$

The tangents from the vertices to the ρ circles are

$$\left.\begin{aligned} &A(\rho_1) = \rho_1 \cot \tfrac{1}{2} A = \frac{-\rho_1 d_4}{c_1};\ B(\rho_2) = \frac{-\rho_2 d_4}{c_2}; \\ &C(\rho_3) = \frac{-\rho_3 d_4}{c_3};\ D(\rho_1) = \frac{d_1 \rho_1}{c_4}; \\ &D(\rho_2) = \frac{d_2 \rho_2}{c_4};\ D(\rho_3) = \frac{d_3 \rho_3}{c_4}. \end{aligned}\right\} \quad \ldots (18)$$

$$AR = \frac{1}{c_2} + d_4 \left(\frac{\rho_2}{c_2} - \frac{\rho_1}{c_1}\right) \quad \ldots (19)$$

$$R(\rho_2) = Q(\rho_3) = \frac{1}{c_1}(1 + d_4\rho_1)$$

$$R(\rho_1) = P(\rho_3) = \frac{1}{c_2}(1 + d_4\rho_2)$$

$$Q(\rho_1) = P(\rho_2) = \frac{1}{c_3}(1 + d_4\rho_3) \quad \ldots (20)$$

5. *Number of solutions of the problem.*

The curvatures c_4, c_4' of the two circles, centres D, D_1, touching the circles c_1, c_2, c_3 are given by

$$c_4 = c_1 + c_2 + c_3 + 2\sqrt{c_1c_2 + c_2c_3 + c_3c_1}$$

$$c_4' = c_1 + c_2 + c_3 - 2\sqrt{c_1c_2 + c_2c_3 + c_3c_1} \quad \ldots (21)$$

Both the curvatures are of the same sign and therefore both the points D, D_1 satisfy the conditions of the problem if

$$c_1 + c_2 + c_3 > 2\sqrt{c_1c_2 + c_2c_3 + c_3c_1},$$

i.e., $$c_1^2 + c_2^2 + c_3^2 > 2(c_1c_2 + c_2c_3 + c_3c_1). \quad \ldots (22)$$

That both the curvatures can be of the same sign is seen from the fact that the two circles, touching (Dc_4), (Bc_2), (Cc_3) have external contact. Thus two solutions or positions of D are obtained for triangles whose sides satisfy the inequality $(2s = a + b + c)$

$$5s^4 + 4sabc + (bc + ca + ab)^2 > 6s^2(bc + ca + ab). \quad \ldots (23)$$

6. ***Two tetrads of circles in singular position.***

The tetrads of circles

$$(Ac_1),\ (Bc_2),\ Cc_3),\ (Dc_4)$$
$$(A'd_1),\ (B'd_2),\ (C'd_3),\ (D'd_4)$$

are so related that a circle of one tetrad cuts orthogonally the three non-corresponding circles of the other and the circles of each tetrad are mutually tangential.

N. Durairajan.

India's Contribution to the Theory of Indeterminate Equations of the First Degree.

While commenting on the elder Aryabhata's rule for the solution of indeterminate equations of the form $Ax - By = C$, Mr. G. R. Kaye writes[1]: "Even a cursory examination of the mathematics of our author will convince any one familiar with the subject that Aryabhata was not the inventor of the method under consideration; and a closer investigation establishes this conclusion beyond all doubt." Different writers have shown that Mr. Kaye's conclusions are, in many cases, unsound and that his methods of investigation are open to serious objections.[2] The object of this paper is to trace the development of the theory of indeterminate equations of the first degree in India and to show thereby that the above remark of Mr. Kaye cannot be supported by facts, and that the Indian treatment of indeterminate equations of the first degree was original and not influenced by Chinese or Greek writers. We shall, therefore, begin by giving an account of the Chinese and Greek contributions to this subject before the 13th century A.D. when mathematics began to decline in India.

Chinese Contribution.

So far as our present knowledge goes, the first author to record a problem leading to indeterminate equations of the first degree is Sun-Tsŭ, a Chinese mathematician of the first century A.D. The problem occurs in his *Suan-ching* (*Arithmetical Classic*) which in the opinion of Tai Chêng and others, was composed after 65 A.D. when Buddhism was introduced into China.[3] Mr. Y. Mikami states the problem as follows:

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1 *Journal of the Asiatic Society of Bengal* (hereafter referred to as JASB), Vol. IV (1908), p. 135

2 Nalinbihari Mitra in the *Modern Review* (Calcutta), Vols. XVII—XIX; Narendrakumar Mazumdar in the *Bulletin of the Calcutta Mathematical Society* (hereafter referred to as the *Bulletin*), Vol. III (1911-12); Bibhutibhusan Datta in the *Bulletin*, Vol. XVII (1926); and in the JASB, Vol. XXII (1926); Saradakanta Ganguli in the *Journal of the Bihar and Orissa Research Society* for March, 1926; in the *Bulletin*, Vols. XVIII and XIX; in the *American Math. Monthly*, Vols- 34 and 37 and in the *Isis*, XII (1929).

3 Y. Mikami: *The Development of Mathematics in China and Japan*, 1913, pp. 25 and 26. (Hereafter referred to as Mikami: *Development*).

"There are certain things whose number is unknown. Repeatedly (*i.e.*, successively) divided by 3, the remainder is 2; by 5 the remainder is 3; and by 7 the remainder is 2. What will be the number?"[4]

This problem leads to the indeterminate equations

$$n = 3x + 2 = 5y + 3 = 7z + 2,$$

the general solution of which is

$$n = 3 . 5 . 7i + 23, \quad x = 5 . 7i + 7, \quad y = 3 . 7i + 4, \quad z = 3 . 5i + 3,$$

where i is zero or any positive integer.

Sun-Tsŭ gives only one solution and states his method as follows:

"The remainder given by the divisor 3 is 2, and so take 140. The remainder corresponding to the divisor 5 is 3, and so take 63. The remainder corresponding to the divisor 7 is 2, and so take 30. The sum of these numbers is 233. Subtracting $210 = (3 . 5 . 7) \times 2$, we obtain the answer 23."

Sun-Tsŭ does not stop here. He also gives his arguments for taking the above numbers (namely, 140, 63, 30) and thus indicates a general line of procedure which is applicable to other similar cases. For, he adds:

"In general, take 70 when the remainder of the division by 3 is 1; take 21 when the remainder of the division by 5 is 1; and take 15 when the remainder of the division by 7 is 1. When the sum of these numbers is above 105, subtract 105 before we get the answer."

The general problem and its solution suggested here are as follows:

If an integer N, when successively divided by the integers $d_1, d_2, d_3, \ldots d_n$, leave the remainders $r_1, r_2, r_3, \ldots r_n$ respectively, then

$$N = D_1k_1r_1 + D_2k_2r_2 + D_3k_3r_3 + \ldots + D_mk_mr_m + \ldots + D_nk_nr_n,$$

where $D_m = (d_1d_2d_3 \ldots d_n) / d_m$, and k_m has a suitable value.

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[4] *Ibid.* p. 32.

The least value of N is the remainder left on dividing the right-hand member by the product of all the given divisors.

Since d_m is a factor of D_p for all values of p from 1 to n with the exception of m, the remainder left on dividing $D_m k_m r_m$ by d_m is the same as the remainder left on dividing N by d_m. But this remainder is r_m by the problem. Therefore, $D_m k_m$, when divided by d_m, gives 1 as the remainder. The solution of the general problem, therefore, depends on the solution of the equation

$$\frac{D_m k_m - 1}{d_m} = \text{an integer},$$

that is, on the solution of the indeterminate equation

$$D_m x = d_m y + 1.$$

The value of k_m or x cannot be easily found except in simple cases like the one considered by Sun-Tsŭ who has found the values of k_1, k_2, k_3 by inspection only and has indicated no method of finding their values when the divisors are not convenient numbers Hence, in the absence of fresh evidence to the contrary, the credit of discovering or of suggesting a solution of an equation of the form $Ax - By = 1$ cannot go to Sun-Tsŭ. But to him certainly belongs the honour of being the first to suggest a simple and elegant method of solution of the general problem of which he gives a particular case, when a solution of the equation $Ax - By = 1$ is already known.

The next Chinese scholar to deal with indeterminate equations of the type $Ax - By = C$ was a Buddhist priest and astronomer, I-hsing by name He was born in the year 683 A.D., and died in 727 A..D. For the construction of his calendrical system which he was ordered in 721 A.D. to compose, he contrived his *t'ai-yen* method which, we are told, was "a consideration in the indeterminate analysis"[5] As a large number of the numbers dealt with in the construction of a calendar must have been inconvenient, the method of inspection, employed by Sun-Tsŭ, was of no avail. I-hsing, therefore, must have been aware of a general solution of the equation $Ax - By = C$. What his method was we do not know. As he was versed in Sanskrit[6] and was posterior to

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[5] Mikami; *Development*, p. 50, [6] *Ibid*, p. 60,

the elder Āryabhaṭa by more than two centuries and to Brahmagupta by about a century, it is not beyond the bounds of possibility that he came to know the elder Āryabhaṭa's method of solution of the equation $Ax - By = C$ with or without the improvements effected by Brahmagupta. Or it may be that he was the inventor of the method which was explained more than five centuries later by Ch'in Chiu-Shao in his *Su-shu Chiu-chang* (Nine Sections of Mathematics) which was written in the year 1247 A.D. For, Ch'in says in his preface that the method was developed in the hands of calendar-makers or astronomers [7] It may be noted here that this method is different from the Indian methods all of which grew, as we shall see below, out of the elder Āryabhaṭa's method.

Greek Contribution.

The first Greek writer to record an example on linear indeterminate analysis is Nicomachus of Gerasa (c. 100 A.D.) who gives Sun-Tsŭ's problem as well as his solution.[8] Diophantus (c. 250 A.D.) did not pay any attention to indeterminate equations of the first degree. "For, as Nesselmann observes, while with indeterminate quadratic equations the object is to obtain a *rational* result, the whole point in solving indeterminate simple equations is to obtain a result in *integral* numbers. But Diophantus does not exclude fractional solutions, and he has therefore only to see that his results are *positive*, which is of course easy. Indeterminate equations of the first degree would therefore, from Diophantus' point of view, have no particular significance."[9] The *Anthology* (c. 500 A.D.) contains two problems leading to indeterminate equations of the first degree which admit of an infinite number of solutions in positive integers.[10] The first problem leads to the equation $x = 4y$ where y is not less than 2. The second problem leads to three equations involving four unknown quantities :

$$x + y = x_1 + y_1,$$
$$x = 2y_1,$$
$$x_1 = 3y.$$

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[7] *Ibid*, p 65.

[8] L. E. Dickson : *Theory of Numbers*, Vol. II, p. 58.

[9] Heath : *Diophantus of Alexandria*, 2nd edn., p 67.

[10] Heath : *History of Greek Mathematics*, Vol. II, p. 443 ; D. E. Smith ; *History of Mathematics*, Vol. II (1925), p. 584.

the general solution of which was easily obtained by taking $y = k$. The general solution of either of these two problems requires no theory of indeterminate equations of the first degree. Hence Greek contribution to this theory is practically nil. Thus there is no justification, whatsoever, for Mr. Kaye's dogmatic statement that "there is no doubt that they (*i.e.*, the Greeks) were able to manipulate indeterminates of the first degree in the manner indicated in the rule of Āryabhaṭa."[11]

India's Contribution.

We have seen that the Greeks contributed nothing to the theory of indeterminate equations of the first degree, In the first century A.D. the Chinese scholar Sun-Tsŭ made a very important contribution ; but it could not be utilised except in simple cases, as a general solution of the equation $Ax - By = 1$ was not known in China earlier than the eighth century. The honour of being the first to give a general solution of the equation $Ax - By = C$ of which the above equation is a particular case must, therefore, go to the elder Āryabhaṭa (*b.* 476 A.D.) whose rule for finding a number which leaves given remainders on being divided by *two* given divisors contains the general solution. Āryabhaṭa's problem in indeterminate analysis appears to be exactly similar to the one given by Sun-Tsŭ. Āryabhaṭa considers only *two* divisors, while Sun-Tsŭ contemplates any number of divisors. This difference may, at first sight, appear to be of no importance. But it is fundamental. Accordingly Āryabhaṭa's solution cannot be extended so as to give a solution of Sun-Tsŭ's problem. Yet the latter solution depends on the former. Whether Āryabhaṭa's problem was suggested by Sun-Tsŭ's problem or was of independent origin or both came out of the same third source we cannot say But it is certain that Āryabhaṭa was not aware of Sun Tsŭ's solution of his problem. If the fact had been otherwise, Āryabhaṭa would have given not only an elegant solution of his problem after Sun-Tsŭ but also a solution of the equation $Ax - By = 1$ (which received a separate consideration at the hands of Brahmagupta and some of his successors) on which such a solution of his problem depends As the solution of the equation $Ax - By = C$ follows at once from that of the equation $Ax - By = 1$, the former equation would not have, in that case, received a special

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[11] JASB, Vol. IV (1908), p. 137.

treatment at the hands of Āryabhaṭa to the entire exclusion of the latter.

Āryabhaṭa's rule mentioned above is as follows:

अधिकाग्रभागहारं छिन्द्यादूनाग्रभागहारेण ।
शेषपरस्परभक्तं मतिगुणमग्रान्तरे क्षिप्तम् ॥
अधउपरिगुणितमन्त्ययुगूनाग्रच्छेदभाजिते शेषम् ।
अधिकाग्रच्छेदगुणं द्विच्छेदाग्रमधिकाग्रयुतम् ॥

[अग्र = remainder. मति = any assumed number. द्विच्छेदाग्रम् = the number corresponding to the two given divisors and the two given remainders.]

These stanzas are elliptical as observed by Prof. P.C. Sen Gupta.[12] Rodet[13] and Mr. Kaye[14] have utterly failed to interpret it. Prof. Sen Gupta's interpretation of the stanzas is based on Brahmagupta's rule which is not the same as Āryabhaṭa's rule, as we shall see hereafter. For the same reason the interpretation given by the Sanskrit commentator Parameswara cannot be accepted as correct. Although Prof. N. K. Mazumdar has not offered any interpretation, he is very near the truth when he observes: "The technical terms of this rule may be best understood by considering the general problem, which most probably gave rise to it"[15] I would like to replace the words " the general problem" by the words "particular (numerical) cases of the general problem." The present writer has arrived at an interpretation by working a numerical example by the method of successive reduction of the co-efficients leading finally to an equation whose general solution can be found by inspection.[16] This method would naturally suggest itself to one familiar with the first four simple rules and with the solution of easy simultaneous determinate equations of the first degree. The name *kuttakavidhi* or method of *kuttaka* (a cutter or pulveriser) given by Indian mathematicians to their method of solution of equations of the form $Ax - By = C$

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[12] The Āryabhaṭiyam *Journal of the Department of Letters* (Calcutta University), reprint, p. 27.

[13] *Journal Asiatique* XIII (1879), pp. 431—34.

[14] JASB, IV (1908), p. 134.

[15] *Bulletin*, Vol. III (1911-12), p. 11;

[16] *Bulletin*. Vol. XIX, No. .

shows that they employed the method of successive reduction of the co-efficients. As the *kuttaka*, by successive operation, reduces the size of an object to the desired extent, so the Indian method, by repeated operation, reduces the size of the given equation so as to be easily solvable by inspection. Let us here solve the general problem by this method in order to show that it also leads to the same interpretation as given by the present writer.

The general problem may be stated thus :

To find a number N which will give the remainders R_1 and R_2 on being divided by A and B respectively, A, B, R_1, R_2 being positive integers.

Let the quotients corresponding to the divisors A and B be x and y respectively. Then

$$N = Ax + R_1 \text{ and } N = By + R_2. \qquad \dots \quad (1)$$

To find N it is necessary to solve the equation,

$$Ax + R_1 = By + R_2. \qquad \dots \quad (2)$$

Let R_2 be greater than R_1 and let us write C for $R_2 - R_1$. Then this equation may be written as

$$Ax = By + C. \qquad \dots \quad (3)$$

Let us now explain some terms which we shall have occasion to use in connection with an equation of this type. When an indeterminate equation is reduced to the form of (3) in which only one term, and that involving an unknown quantity, occurs on one side, the co-efficient of that unknown quantity [*e.g.*, A in (3)] is called the *divisor* as it divides the right-hand member of the equation without a remainder. The co-efficient of the other unknown quantity on the other side is called the *dividend*. In equation (3) B is called the dividend. The dividend is sometimes called the *multiplier*. The term independent of the unknown quantities is called the *kṣepa* (what is thrown into or away from, *i.e.*, what is added to or subtracted from) which, therefore, may be positive or negative. In equation (3) C is a *kṣepa*. For the sake of convenience we shall give the name *dividend-side* or *divisor-side* to a side of an equation of the form (3) according as the absolute term does or does not occur on the side. Therefore, by transposition of the abso-

lute term from the dividend-side to the divisor-side the former can be converted into the latter and *vice versa.*

Case I.—Let B be greater than A and equal to $AQ_1 + r_1$. Then equation (3) can be solved if we can solve

$$A(x - Q_1 y) = r_1 y + C.$$

As the right-hand side is a positive integer, $x - Q_1 y$ is a positive integer. Hence this equation may be written as

$$Ax_1 = r_1 y + C \qquad \dots \quad (4)$$

where $$x_1 = x - Q_1 y.$$

Thus we see that the solution of equation (3) can be made to depend on the solution of equation (4) in which the co-efficient of y (*i.e.*, the co-efficient on the dividend-side) is smaller than in (3). As A is greater than r_1, we can, by repeating the process, make the solution of equation (4) depend on the solution of another equation in which the co-efficient of x_1 (*i.e*, the co-efficient on the divisor-side) is less than in (4). Thus, if $A = Q_2 r_1 + r_2$, equation (4) can be solved if we can solve

$$r_2 x_1 = r_1(y - Q_2 x_1) + C$$

or $$r_2 x_1 = r_1 y_1 + C, \qquad \dots \quad (5)$$

where $$y_1 = y - Q_2 x_1.$$

Proceeding in this way we may alternately reduce the co-efficients on the two sides until one of the co-efficients becomes unity. So let $r_1 = Q_3 r_2 + r_3$, then equation (5) can be solved if we can solve

$$r_2(x_1 - Q_3 y_1) = r_3 y_1 + C$$

or $$r_2 x_2 = r_3 y_1 + C, \qquad \dots \quad (6)$$

where $$x_2 = x_1 - Q_3 y_1.$$

Next let $r_2 = Q_4 r_3 + r_4$, then equation (6) can be solved if we can solve

$$r_4 x_2 = r_3(y_1 - Q_4 x_2) + C$$

or, $$r_4 x_2 = r_3 y_2 + C, \qquad \dots \quad (7)$$

where $$y_2 = y_1 - Q_4 x_2.$$

Since the remainder r_4 has been obtained as the result of division by r_3, r_4 is less than r_3. Hence, if, at this stage, one of the co-efficients becomes unity, it must be r_4. Let us suppose, for the sake of simplicity, that $r_4 = 1$. Then equation (7) becomes

$$x_2 = r_3 y_2 + C. \qquad \dots \quad (8)$$

The general solution of this equation is obtained by taking $y_2 = t$ where t is zero or any positive integer. Aryabhaṭa gives the name ***mati*** to this t. Since equation (8) can be solved by inspection, the solution of the original equation follows from the following set of equations:

(9)	$x = Q_1 y + x_1$	from (4)
(10)	$y = Q_2 x_1 + y_1$	from (5)
(11)	$x_1 = Q_3 y_1 + x_2$	from (6)
(12)	$y_1 = Q_4 x_2 + y_2$	from (7)
(13)	$x_2 = r_3 t + C$	from (8)
(14)	$y_2 = t.$	

The last quotient of mutual division is Q_4 or r_3 according as the last remainder is taken to be 1 or 0. As r_3 occurs below the quotients Q_1, Q_2, Q_3, Q_4, we regard r_3 as the last quotient. Hence, in this case the operations of mutual division are continued until the remainder is zero. Thus we notice that in the set of equations (9) — (14) the quotients Q_1, Q_2, Q_3, Q_4, r_3 of mutual division occur in order in the same vertical column, the first quotient Q_1 being at the top and the last quotient r_3 being followed by the assumed integer t. Here we have begun by dividing B (***adhikāgrabhāgahāra***, the given divisor corresponding to the greater remainder R_2). by A (***ūnāgrabhāgahāra***, *i.e.*, the divisor corresponding to the smaller remainder R_1). Hence Aryabhaṭa says: "***Adhikāgrabhāgahāram chindyāt ūnāgrabhāgahāreṇa.***" (*i.e.*, one should divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder). To find x or y from the set of equations (9)—(14) we have to start from below by taking $x_2 = r_3 t + C$, *i.e.*, by multiplying the last quotient r_3 of mutual division by the assumed integer (***mati***) t and adding the product to C (the difference of the given remainders). Aryabhaṭa expresses these operations in the words "***Śeṣaparas-parabhaktaṃ matiguṇaṃ agrāntare kṣiptam***". It is easy to see that of the remaining unknown numbers x, y, x_1, y_1, each is obtained by multiplying by the corresponding quotient the number next below it and adding

to the product the number just below these two numbers. This law is expressed by Āryabhaṭa in the words "*adha uparigunitam antyayuk.*" This law may be extended so as to find x_2 by setting down C (the difference between the given remainders) below t.

The values of x and y thus obtained will be respectively of the forms $Bmi + \alpha$ and $Ami + \beta$ where α and β are the least integral roots of the equation $Ax - By = C$, *i.e*, of the equation

$$Ax + R_1 = By + R_2.$$

For,
$$A\alpha + R_1 = B\beta + R_2.$$

Also
$$ABmi = BAmi.$$

Therefore
$$A(Bmi + \alpha) + R_1 = B(Ami + \beta) + R_2.$$

As the least value of N is $B\beta + R_2$, it is obtained by adding the greater remainder R_2 to the product of the corresponding divisor B and the remainder left on dividing by A (the divisor corresponding to the smaller remainder) the lower of the two numbers obtained finally from the set of equations (9)—(14), which is the general value of y and is of the form $Ami + \beta$. Hence Āryabhaṭa says: '*Ūnāgracchedabhājite s'esam adhikāgracchedagunam dvicchedāgram adhikāgrayutam.*"

Thus we arrive at Āryabhaṭa's rule quoted above.

Case II.—Let A be greater than B.

Then $Q_1 = 0$, $r_1 = B_1$ and $x_1 = x$; and equation 9) becomes $x = x_1$. Therefore to solve the equation $Ax - By = C$ we must either reject equation (9) and write x for x_1 in (10) and (11) or retain it in the form $x = x_1$. In the former case the order of the equations giving x and y would be the reverse of that in Case I; in the latter case the order would not be disturbed. For the purpose of a general rule it is advisable to stick to the same order. Hence we retain the equation (9) in the form $x = x_1$ which may be written in the form $x = 0 . y + x_1$ in order to satisfy the law noted above of obtaining the unknown numbers x, y, x_1, y_1. The divisor corresponding to the greater remainder being, in the present case, smaller than the other divisor, it gives the quotient 0 when divided by the latter; and this quotient is the co-efficient of y in $x = 0 . y + x_1$. Hence, in this case also, the rule requires that one should begin by dividing the divisor corresponding to the greater remainder by the other given divisor.

Thus we arrive at the following interpretation of Āryabl
quoted above :

"To find a number which, when divided by two giver
leaves given remainders, divide the divisor corresponding to t
remainder by the divisor corresponding to the smaller. The
this operation of division should then be divided by the remai
by this operation. This process of mutual division should be
so long as the remainder does not vanish. The quotients c
division should be set down, one below another in a vertical li
order in which they are obtained. Set down any assumed integ
these quotients and the difference between the given remaind
it. Then multiply the last quotient of mutual division by the
integer and add to the product the difference between the
mainders. Continue this process of multiplying a lower numbe
one just above it and adding to the product the number just und
each case the lowest number added should be rejected just
operation of addition so that the next upper number in the
column may, in its turn, be the lowest in position. Finally two
will be obtained which are the quotients of division of the
number by the given divisors, the upper number corresponding to
smaller given remainder and the lower number to the greater given remainder. The lower of these two numbers should be divided by the given divisor corresponding to the smaller of the given remainders. The remainder thus obtained, being multiplied by the given divisor corresponding to the greater of the given remainders and then increased by the greater given remainder, gives the least number answering to the two given divisors and to the two given remainders. The implication is that the least number satisfying the given conditions can also be obtained by multiplying the remainder, obtained as the result of division of the upper number by the divisor corresponding to the greater given remainder, by the divisor corresponding to smaller given remainder and then adding the smaller given remainder to the product.[17]

SARADAKANTA GANGULI.

(*To be continued.*)

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[17] S. K. Ganguly: *The source of the Indian solution of the so-called Pellian equation*, (*Bulletin*, Vol XIX, No. 4).

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Solutions.

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Question 381.

Bmi + aRAMASWAMY AIYER, M.A.):—O is the circum-centre of a of the eq BC, PQ is any diameter of an inscribed conic. If D be a BC such that the angles AOD and POQ have the same , show that the feet of the perpendiculars from O on the sides For, adrilateral APDQ lie on a circle touching the nine-points circle.

Also

Therefor

Solution by T. R. Raghava Sastri.

dently a conic can be inscribed in the quadrilateral APDQ, As) as a focus. For, to the involution pencil formed by the pairs greater DP, OQ and OA, OD, the circular lines through O also belong, the rem s given that the above pairs have common bisectors.

smaller

the set e auxiliary circle of that conic therefore is the circle through the the forr the perpendiculars from O on the sides of the quadrilateral. s'exam er the auxiliary circle of the in-conic of the triangle ABC with O as focus is the nine-points circle

Now reciprocating with respect to O, the triangle ABC reciprocates into the triangle LMN for which O is the in-centre and the in-conic of ABC for which PQ is a diameter, reciprocates into a circum-conic of LMN with the points P, Q reciprocating into tangents at the extremities of a chord through O. The in-conic above-mentioned with O as focus corresponds to the circum-circle of LMN. The conic touching the sides of APDQ with O as focus reciprocates into a circle passing through the intersections of the tangents corresponding to P, Q with the side MN and with a proper line through L so as to make the quadrilateral thus obtained cyclic. These two circles are evidently the inverses of the two auxiliary circles in the circte of reciprocation with centre O. To show therefore that these inverses touch, taking LMN as the reference triangle, and using a real co-ordinates, the co-ordinates of O are l, m, n, (where l, m, n are the sides of the triangle LMN). Let the equation of the circum-conic be

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0.$$

The equations of the two tangents to it are of the form

$$\frac{ax}{x_1^2}+\frac{by}{y_1^2}+\frac{cz}{z_1^2}=0 \text{ and } \frac{ax}{x_2^2}+\frac{by}{y_2^2}+\frac{cz}{z_2^2}=0,$$

As these tangents are the tangents at the ends of the chord through O the condition is

$$\frac{al}{x_1x_2}+\frac{bm}{y_1y_2}+\frac{cn}{z_1z_2}=0. \qquad \ldots \quad (1)$$

The equation of the circle inverse to the circle through the feet of the perpendiculars from O to the sides of APDQ will be of the form

$$\left(\frac{ax}{x_1^2}+\frac{by}{y_1^2}+\frac{cz}{z_1^2}\right)\left(\frac{ax}{x_2^2}+\frac{by}{y_2^2}+\frac{cz}{z_2^2}\right)+x\,(py+qz)=0$$

from which it is evident that the length of the tangents from the vertices of LMN must be proportional to

$$\frac{a}{x_1x_2},\ \frac{b}{y_1y_2},\ \frac{c}{z_1z^2}$$

and therefore this circle will touch the circum-circle of LMN as the condition (1) is satisfied. Hence the result.

Question 420.

(K. J. SANJANA, M.A.):—In the plane of three fixed points A, B, C, a fixed straight line is drawn through A, and a variable circle touches it at that point. Prove that the locus of the intersection of the line joining B to the centre of the circle and the polar of C with regard to the circle is a conic. (Geometrical proof preferred). Show that Question 390 is a particular case of this theorem.

If BM, CN are perpendicular to the fixed straight line, examine the locus when AM . AN = — BM . CN.

Solution by T. R. Raghava Sastri.

If O is the centre of a circle touching the fixed line AR at the fixed point A, O moves on the perpendicular AO at A to the fixed line. If

LP is the polar of C for the circle O, since $OA^2 = OL.OC$, therefore the circle through C, touching OA at A, passes through L. Hence LP passes through S, a fixed point on the last circle being diametrically opposite to C on it. Since the rays BO and CO trace the same range on AO, and LP is perpendicular to CO, the rays BO and SP are corresponding rays of two homographic pencils. Hence the locus of P is a conic passing through B and S and evidently through A also.

When $BM.CN = -AM.AN$, $\triangle BMA$ is similar to $\triangle CNA$, hence angles at A are complementary. Hence $B\hat{A}C$ is a right angle.

But $C\hat{A}S$ is a right angle. So, BAS is a straight line.

Hence the conic breaks up into two straight lines, one of which is the line BAS itself. Hence the locus of P is a straight line. This also follows from the fact that in the two homographic pencils at B and S, the two corresponding rays BA and AS coincide.

Question 390 is only a particular case of this. For, let the variable circle in that question be the circle O above and let the centre of the fixed circle be B. Then the centre of the orthogonal circle lies on BO evidently. PQ clearly passes through a fixed point C on AR. The pole of P, Q which is the centre of the orthogonal circle through P, Q must lie therefore on the polar of C with regard to the circle O. Hence the centre is the intersection of BO with the polar of C.

Partially solved analytically by S. P. Ranganathachar.

[*Question 390.* A variable circle Ctouches a given straight line at a given point and cuts a given circle in two points P, Q. Find the locus of the centres of circles cutting the circle C orthogonally at P, Q.]

Question 1511.

(S. N. KUMARASWAMI) :—In a triangle ABC, M is the middle point of BC, K is the point of contact of the in-circle with BC, AD the altitude, I the in-centre, O the ortho-centre and Q the middle point of AO. Show geometrically that $MK.KD + IK.QD = Rr$.

Solution by A. Ranganatha Rao, B. Ramamurti, K. V. Krishna Sastri, T. S. Venkataraman, S. Rajan, K. V. Vedantam and M. V. Seshadri.

MQD is a rt.-angled triangle inscribed in the nine-points circle.

∴ The middle point N of MQ must be the nine-points cent Drop NP perpendicular to BC. Then

$$IN^2 = (IK - NP)^2 + PK^2.$$

$$\therefore \quad \left(\frac{R}{2} - r\right)^2 = \left(r - \frac{1}{2}QD\right)^2 + PK^2, \text{ (by Feuerbach's Theorem.)}$$

$$\therefore \quad \left(\frac{R}{2}\right)^2 - \left(\frac{1}{2}QD\right)^2 - PK^2 = r\,.\,R - r.QD.$$

$$PM^2 - PK^2 = rR - IK\,.\,QD.$$

$$\therefore \quad MK\,.\,KD = rR - IK\,.\,QD.$$

Question 1512.

(A. A. KRISHNASWAMI AIYANGAR):—If the sides of a triangle are denoted by the equation $f(x, y) = 0$ in rectangular co-ordinates, show that the radical axis of the circum-circle and the nine-points circle of the triangle is given by

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Solution by S. P. Ranganathachar G. V. Krishnaswami, R. Kothandaraman, Hukamchand, B. Ramamurthi, K. V. Vedantam, S. Ramaswami Aiyar and Budhram Aggarwal.

Let the equations of the sides BC, CA and AB of a triangle ABC be

$$L_1 \equiv a_1x + b_1y + c_1 = 0$$

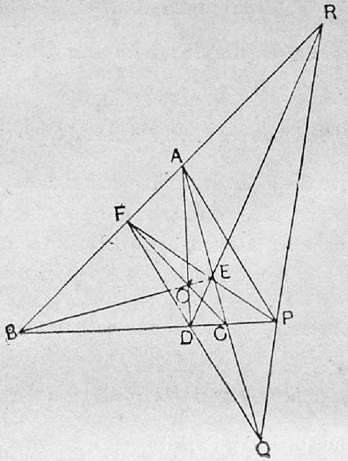
$$L_2 \equiv a_2x + b_2y + c_2 = 0$$

$$L_3 \equiv a_3x + b_3y + c_3 = 0,$$

Then $f(x, y) = L_1L_2L_3 = 0$ is the equation to the sides.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \text{ becomes}$$

$$L \equiv (a_2a_3 + b_2b_3)\,L_1 + (a_3a_1 + b_3b_1)\,L_2 + (a_1a_2 + b_1b_2)\,L_3 = 0.$$



We are to prove that this is the equation to the radical axis of the circum-circle and the nine-points circle.

In the figure, AD, BE, CF are the altitudes of the triangle ; O, the ortho-centre ; EF, FD, DE meet the opposite sides in P, Q, R respectively. It is known that PQR is a line.

Also, as PB . PC = PE . PF, etc., PQR is the radical axis of the circum-circle and the nine-points circle.

The equation of AD is $L_2 + \lambda L_3 = 0$

with $$a_1(a_2 + \lambda a_3) + b_1(b_2 + \lambda b_3) = 0,$$

i.e., $$(a_1a_3 + b_1b_3)\,L_2 - (a_1a_2 + b_1b_2)\,L_3 = 0$$

AP being the harmonic conjugate of AD has for its equation

$$(a_1a_3 + b_1b_3)\,L_2 + (a_1a_2 + b_1b_2)\,L_3 = 0.$$

Thus L passes through P. Similarly it can be proved that it passes through Q and R. Or L is the equation to PQR which is the radical axis of the circum-circle and the nine-points circle.

Question 1516.

(N. DURAIRAJAN):—Show that the locus of the node of a variable cubic having three fixed asymptotes and having orthogonal nodal tangents, is the radical axis of the circum-circle and the polar circle of the triangle formed by the asymptotes.

Solution by T. R. Raghava Sastri.

We shall prove the following more general theorem : " The locus of a point such that its polar conic with respect to a cubic with three given asymptotes, is a rectangular hyperbola, is the radical axis of the polar and circum-circles of the triangle formed by the asymptotes."

Taking the given asymptotes as the base triangle, the areal equation of the cubic is

$$xyz + (x + y + z)^2 (lx + my + nz) = 0.$$

The polar conic of (x_1, y_1, z_1) is

$$(x_1yz + y_1zx + z_1xy) + 2(x + y + z)(x_1 + y_1 + z_1)(lx + my + nz) + (x + y + z)^2 (lx_1 + my_1 + nz_1) = 0$$

which is similar to the conic

$$x_1yz + y_1zx + z_1xy = 0.$$

The condition for this to be a rectangular hyperbola is

$$x_1 \cot A + y_1 \cot B + z_1 \cot C = 0.$$

$\therefore$ the locus of $(x_1\ y_1\ z_1)$ is

$$x \cot A + y \cot B + z \cot C = 0$$

which is the radical axis of the polar and circum-circles of the base triangle.

Remark. In Question 1512, if the cubic is taken in cartesians as

$$f(x, y) + lx + my + n = 0,$$

the condition for the polar conic to be a rectangular hyperbola leads to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Let BC, CA, AB be the tangents at the points θ_1, θ_2, θ_3 and P', Q', R' the mid. points of these tangent-chords.

$OP' = OQ' = OR' = b$ (*i.e.*) O is the circum-centre of the triangle $P'Q'R'$.

Again $$\angle P'OQ' = \theta_2 - \theta_1.$$

$$\therefore \qquad \angle R' = \frac{\theta_2 - \theta_1}{2}$$

$$\angle C = \psi_1 - \psi_2 = \frac{\theta_2 - \theta_1}{2} = \angle R'.$$

Similarly $\angle$s B and A are respectively equal to $\angle$s Q' and P' or the triangle $P'Q'R'$ is inscribed in the triangle ABC and directly similar to it. Hence the circum-centre of triangle ABC coincides with the ortho-centre of triangle $P'Q'R'$.

(Page 132, Q. 5, Lachlan: *Modern Pure Geometry*).

Questions 1521, 1522, 1523.

Remarks by A. A. Krishnaswamy Aiyangar.

Questions 1521, 1522 are completely discussed by P. A. MacMahon in the *Messenger of Mathematics*, Vol. XII, pp. 151—153, (1883) to which the reader is referred for further information. M. Humbert has drawn out numerous consequences of the third tangent property in the *American Journal of Mathematics*, Vol. X, 1888.

Regarding Question 1523, one has only to note that the triangle formed by the middle points of three tangent-chords is similar to the triangle formed by the chords themselves. The theorem of this question is fairly well-known, and I may add that it was kindly communicated to me by Mr. V. Ramaswami Ayyar in 1926.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their Questions.

1629. (T. R. Raghava Sastry):—If lines OL, OM, ON are drawn from a point O on the circum-circle of a triangle ABC meeting the sides in L, M, N and equally inclined to the sides at angle θ measured in the same sense round the triangle, then it is known that L, M, N are collinear. Let us call the line of collinearity a θ-poncelet line of ABC for O.

If two triangles are such that the sides of each are θ-poncelet lines of the other, show that the six sides of the two triangles touch a conic whose centre is the mid. point of the line joining the circum-centres of the two triangles.

1630. (T. R. Raghava Sastry):—If two triangles are in perspective and also orthologique (*i.e.*) the perpendiculars from the angular points of one triangle on the corresponding sides of the other are concurrent, show that the centre of perspective is collinear with the two orthologique centres.

1631. (T. R. Raghava Sastry):—If perpendiculars from a point O in the plane of a triangle ABC, on the sides of the triangle meet the pedal circle of O in L, M, N (not the feet of the perpendiculars on the sides) and if P, Q, R are taken on OL, OM and ON such that L, M, N are the mid. points of OP, OQ, OR,

(1) Show that AP, BQ, CR meet at the twin-point of O for the triangle ABC;

(2) Generally if $\frac{OP}{OL} = \frac{OQ}{OM} = \frac{OR}{ON}$, show that AP, BQ, CR are concurrent at a point on the rectangular hyperbola through ABCO

1632. (R. Vaidyanathaswamy):—ABCD is a cyclic quadrangle. If the pedal line of A with respect to BCD is perpendicular to the Euler line (*i.e.*, join of circum-centre and centroid) of BCD, prove that a similar property holds for each of the vertices.

When this property is possessed by ABCD, prove that the perpendiculars let fall from each vertex on its pedal line with respect to the remaining three concur in a point.

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1 Annales de l'Ecole Normale Superieure, No. 7, July 1932.
2 Annals of Mathematics, Vol. 33, No. 3 (2 copies).
3 Astrophysical Journal, **75**, 5, **76**, 1 and 2.
4 Bulletin of the American Mathematical Society, **38**, 6, 7, 8.
5 Bulletin des Sciences Mathematiques, **56** June 1932.
6 Crelle's Journal, **168**, 2.
7 Current Science, **1**, 1, 2, 3.
8 Jahresbericht der deutschen mathematiker Vereinigung, **41**, 9—12.
9 Japanese Journal of Mathematics, **9**, 1.
10 Journal of the Faculty of Science, **1**, 3.
11 Mathematische Annalen, **106**, 5, **107**, 1, 2.
12 Monthly Notices of the Royal Astronomical Society, **92**, 7, 8.
13 Philosophical Magazine, **14**, 89, 90.
14 Philosophical Transactions of the Royal Society, **231**, pp. 55—73.
15 Popular Astronomy, **40**, 7.
16 Proceedings of the Cambridge Philosophical Society, **28**, 3.
17 Proceedings of the London Mathematical Society, **34**, 1, 2, 3 (3 copies each).
18 Proceedings of the Physico-Mathematical Society of Japan, **14**, 7.
19 Proceedings of the Royal Society of London, **137**, 832, 833.
20 Revista Mathematica Hispano, **7**, 3 and 4.
21 Transactions of the Royal Society of South Africa, **20**, 4.

Books.

Cambridge tracts in Mathematics and Mathematical Physics.

1 The Foundations of Differential Geometry, No. 29, by Oswald Veblen and J. H. C. Whitehead.
2 Conformal Representation, No. 28, By C. Caratheodory.
3 Hand-Book of the University of Bombay, 1932.
4 Madras University Calendar, Vol. 1, Parts 1 and 2, 1932-1933.

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